Mar 18

Ex 9.4.1

Q1

$$\int \underbrace{x^2}_{\frac{dv}{dx}} \underbrace{\ln x}_{u}\, dx$$

$u = \ln x$ $\quad \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^2$ $\quad v = \frac{x^3}{3}$

$$= \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx$$

$$= \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C \quad \square$$

Ex.

$$\int \ln x\, dx = \int \underbrace{1}_{\frac{dv}{dx}} \cdot \underbrace{\ln x}_{u}\, dx$$

$u = \ln x$ $\quad \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = 1$ $\quad v = x$

integration by parts

$$= x\ln x - \int x\cdot\frac{1}{x}\,dx$$

$$= x\ln x - \int 1\,dx$$

$$= x\ln x - x + C \quad \square$$

Ex 9.4.1 Q2:

$$\int \underbrace{x}_{u} \underbrace{a^x}_{\frac{dv}{dx}}\, dx$$

$u = x$ $\quad \frac{dv}{dx} = a^x$

$\frac{du}{dx} = 1$ $\quad v = \frac{a^x}{\ln a}$

$= \ldots$ integration by parts using

$= \ldots$

Ex

$\int \ln x\,dx = x\ln x - x + C$

$$\int (\ln x)^2\,dx = \int \underbrace{(\ln x)}_{u}\underbrace{(\ln x)}_{\frac{dv}{dx}}\,dx$$

$u = \ln x$ $\quad \frac{dv}{dx} = \ln x$

$\frac{du}{dx} = \frac{1}{x}$ $\quad v = x\ln x - x$

$$= \ln x\,(x\ln x - x) - \int (x\ln x - x)\cdot\frac{1}{x}\,dx$$

$$= \ln x\,(x\ln x - x) - \int (\ln x - 1)\,dx$$

$$= x\ln x(\ln x - 1) - (x\ln x - x) + x + C$$

$$= \ldots$$

__Definition__: Rational Function: $R(x) = \dfrac{p(x)}{q(x)}$ p, q are polynomials, $q \neq 0$.

A __proper__ rational function: is a rational function with $\deg p(x) < \deg q(x)$

When $\deg q(x) = 1$, $\dfrac{p(x)}{q(x)}$ is proper if $p(x) = c$ is a constant function

__Type__: $R = \dfrac{p(x)}{q(x)} = \dfrac{c}{ax+b}$ $a \neq 0$

__E.g.__, $\displaystyle\int \frac{1}{3x+2}\,dx$ use integration by substitution: $u = 3x+2$, $du = 3dx$

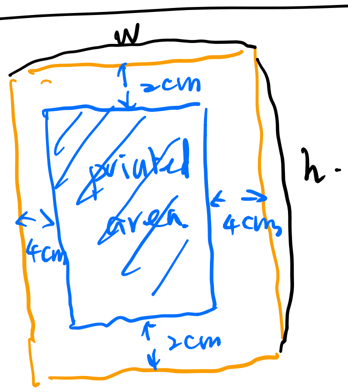
$$= \int \frac{1}{u}\,\frac{du}{3}$$

$$= \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln|3x+2| + C$$

$\square$

__HW4 Q20__



Area of printed material $= 384\ cm^2$

area of poster $= hw$ ← want to minimize.

printed region height $h - 2(2) = h - 4$ (cm)

width $w - 2(4) = w - 8$ (cm)

Area of printed region $= (h-4)(w-8) = 384\ cm^2$

$$hw - 4w - 8h + 32 = 384 \text{ cm}^2$$

$$h(w-8) - 4w = 352 \text{ cm}^2$$

$$h = \frac{352 + 4w}{w-8}$$

area of poster $= hw$

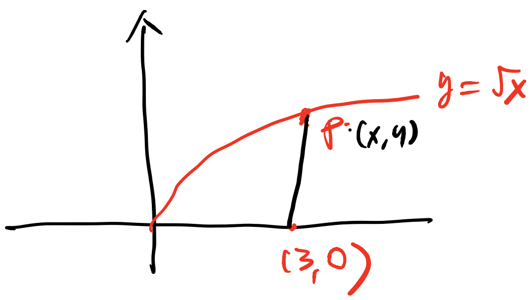
$$= \frac{352 + 4w}{w-8} \cdot w = f(w)$$

now minimize : find abs. minimum

when. $w \geq 8$

$h \geq 4$

Q.17 Find the pt P on the graph of. the function $y = \sqrt{x}$ closest to the pt $(3,0)$



$f =$ distance btwn. $\overset{(x,y)}{P}$ & $(3,0)$

$$= \sqrt{(x-3)^2 + (y-0)^2}$$

$$= \sqrt{(x-3)^2 + y^2}$$

P is on the graph, so, x, y satisfie the relation $y = \sqrt{x}$

$$f(x) = \sqrt{(x-3)^2 + x} \leftarrow \text{ now minimize this function}$$

1. $\int x^3 e^{x^4}\,dx = \frac{1}{4}e^{x^4} + C.$

2. $\int 6x\sqrt{x^2+3}\,dx = 2(x^2+3)^{\frac{3}{2}} + C.$

3. $\int e^x\sqrt{e^x+1}\,dx = \frac{2}{3}(e^x+1)^{\frac{3}{2}} + C.$

let $u = e^x+1$ $du = e^x\,dx$

du

$\int \sqrt{u}\,du = \frac{u^{\frac{3}{2}}}{3/2} + C = \frac{2}{3}(e^x+1)^{3/2} + C$ □

4. $\int (2x-1)(x^2-x)^{100}\,dx = \frac{1}{101}(x^2-x)^{101} + C$

9.4 Integration by Parts

($\Leftrightarrow$ Leibniz rule for products in differential calculus)

Motivation

Let $u(x)$ and $v(x)$ be differentiable functions. By the product rule, we have

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

or

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integrating both sides with respect to x,

$$\begin{aligned}\int u\frac{dv}{dx}\,dx &= \int \frac{d}{dx}(uv)\,dx - \int v\frac{du}{dx}\,dx\\ &= uv - \int v\frac{du}{dx}\,dx\end{aligned}$$

which is

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

or

$$\boxed{\int u\,dv = uv - \int v\,du}$$

formally

Key Idea: Write the integrand as product of $u(x)$ and $v'(x)$, then integrate by parts.

Example 9.4.1. Compute $\int xe^x\,dx.$

$(e^x)'$

apply integration by parts

$u = x$

$e^x = \frac{dv}{dx} \Rightarrow v = e^x$

$\frac{dx}{dx} = 1$

$= xe^x - \int e^x \cdot 1\,dx = xe^x - e^x + C$ □

Solution.

$$\begin{aligned}\int xe^x\,dx &= \int x\,de^x \qquad (u = x, v = e^x)\\ &= xe^x - \int e^x\,dx\\ &= xe^x - e^x + C\end{aligned}$$

Question: What happens if we let $u = e^x$ and $v = \frac{1}{2}x^2$?

$u = e^x$ $\quad \frac{du}{dx} = e^x$

$\frac{dv}{dx} = x$ $\quad v = \frac{x^2}{2}$

$$\begin{aligned}\int \underset{\frac{dv}{dx}}{x}\,\overset{u}{e^x}\,dx &= \int e^x\,d\left(\frac{1}{2}x^2\right)\\ &= \frac{1}{2}x^2e^x - \int \frac{1}{2}x^2\,de^x\\ &= \frac{1}{2}x^2e^x - \int \frac{1}{2}x^2e^x\,dx \qquad \text{(More complicated!)}\end{aligned}$$

■

Example 9.4.2.

take: $\frac{dv}{dx}$ u

$$\begin{aligned}\int x \ln x\,dx &= \int \ln x\,d\left(\frac{1}{2}x^2\right) \qquad \left(u = \ln x, v = \frac{1}{2}x^2\right)\\ &= \frac{1}{2}x^2\ln x - \int \frac{1}{2}x^2\,\cancel{d(\ln x)}\,\frac{1}{x}\,dx\\ &= \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx\\ &= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C \quad \square\end{aligned}$$

$v = \frac{x^2}{2}$ $\quad \frac{du}{dx} = \frac{1}{x}$

Question: What happens if we let $\int x \ln x\,dx = \int x\,d(?)$

$v'(x) = \ln x$, not easy to find v!

Remark. Choose proper u and v such that:

1. it's easy to write the integral as $\int u\,dv$;
2. it simplifies the problem after integration by parts.

Exercise 9.4.1.

1. $\int x^2 \ln x\,dx = \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C$
2. $\int xa^x\,dx = \frac{1}{\ln a}xa^x - \frac{1}{\ln^2 a}a^x + C, \quad (a > 0, a \neq 1)$

Example 9.4.3.

$$\begin{aligned}\int \ln x\, dx &= x\ln x - \int x\, d(\ln x) \qquad (u = \ln x, v = x)\\ &= x\ln x - \int 1\, dx\\ &= x\ln x - x + C\end{aligned}$$

Exercise 9.4.2. $\int \log_a x\, dx = x\log_a x - \dfrac{x}{\ln a} + C$

Hint: either integration by parts directly, or use $\log_a x = \dfrac{\ln x}{\ln a}$.

Example 9.4.4. (Integration by parts twice)

1.

$u = x^2 \qquad \dfrac{dv}{dx} = e^x$

$\dfrac{du}{dx} = 2x \qquad v = e^x$

$$\begin{aligned}\int \underset{\substack{\parallel\\ u}}{x^2}\underset{\substack{\parallel\\ \frac{dv}{dx}}}{e^x}\, dx &= \int x^2\, de^x \quad \text{integration by parts}\\ &= x^2e^x - \int e^x\, dx^2\\ &= x^2e^x - \int 2xe^x\, dx \quad \text{integration by parts (Ex. 9.4.1)}\\ &= x^2e^x - \int 2x\, de^x\\ &= x^2e^x - 2(xe^x - \int e^x\, dx)\\ &= x^2e^x - 2(xe^x - e^x + C)\\ &= x^2e^x - 2xe^x + 2e^x + C'\end{aligned}$$

2.

$u = (\ln x)^2 \qquad \dfrac{dv}{dx} = 1$

$\dfrac{du}{dx} = 2\ln x \cdot \dfrac{1}{x} \qquad v = x$

$$\begin{aligned}\int \ln^2 x\, dx &= x\ln^2 x - \int x\, d(\ln^2 x) \quad \text{integrate by parts}\\ &= x\ln^2 x - \int x \cdot 2\ln x \cdot \frac{1}{\cancel{x}}\, dx\\ &= x\ln^2 x - \int 2\ln x\, dx \quad \text{integration by parts Ex. 9.4.3}\\ &= x\ln^2 x - 2x\ln x + 2\int x\, d(\ln x)\\ &= x\ln^2 x - 2x\ln x + 2x + C\end{aligned}$$

$\underset{\substack{\parallel\\ \frac{dv}{dx}}}{1}\cdot\underset{\substack{\parallel\\ u}}{(\ln x)^2}$

Exercise 9.4.3. $\int (x^2 + 2x + 3)e^x\, dx = (x^2 + 3)e^x + C.$

9.5 Integration of Rational Functions

Rational function:

$$R(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials with $q(x) \neq 0$.

How to integrate $\displaystyle\int \frac{p(x)}{q(x)}\,dx$?

9.5.1 $\boxed{\deg q(x) = 1 : q(x) = ax+b, a \neq 0}$

Let $a \neq 0$. By long division,

proper rational function.

$$\frac{p(x)}{ax+b} \xlongequal{\text{long division}} \underbrace{A(x)}_{\text{polynomial}} + \underbrace{\frac{r}{ax+b}}_{\text{know how to integrate!}},$$

where $A(x)$ is a polynomial and r is a constants.

$$\int \frac{1}{ax+b}\,dx = \int \frac{1}{ax+b} \cdot \frac{1}{a}\,d(ax+b) = \frac{1}{a}\ln|ax+b| + C$$

Example 9.5.1. Evaluate

$$\int \frac{x^2+3x+5}{x+1}\,dx.$$

Solution. By the long division

$$\begin{array}{r|l} & \quad x+2. \\ \hline x+1) & x^2+3x+5 \\ & -x^2 \;\; -x \\ \hline & \qquad 2x+5 \\ & \quad -2x-2 \\ \hline & \qquad\quad 3 \end{array}$$

So,

$$\begin{aligned} \int \frac{x^2+3x+5}{x+1}\,dx &= \int (x+2) + \frac{3}{x+1}\,dx \\ &= \frac{x^2}{2} + 2x + 3\ln|x+1| + C. \end{aligned}$$

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